Discrete Mathematics: Combinatorics and Graph Theory

Exam 1 Solution

Instructions. Solve any 5 questions and state which 5 you would like graded. Write neatly and show your work to receive full credit. You must sign the attendance sheet when returning your booklet. Good luck!

1. Prove or disprove the following:

   (a) For all $m, n \in \mathbb{Z}$, if mn are odd, then m and n are odd.
   Consider the contrapositive: *If m or n are even, then $m \times n$ is even.*
   By definition the product is even in each of the three cases:

      i. m is even and n is odd:
      $\Rightarrow m = 2a,\ n = 2b+1 \Rightarrow m \times n = 2ab + 2a = 2(ab+a) \Rightarrow m \times n$ is even.

      ii. m is even and n is even:
      $\Rightarrow m = 2a,\ n = 2b \Rightarrow m \times n = 2a \times 2b = 4ab = 2(2ab) \Rightarrow m \times n$ is even.

      iii. m is odd and n is even:
      $\Rightarrow m = 2a+1,\ n = 2b \Rightarrow m \times n = 4ab + 2b = 2(2ab+b) \Rightarrow m \times n$ is even.

   (b) There are no integer solutions to the equation $x^2 + 19 = y^2 + 2021$.
   We know that $(x+y)(x-y) = 2002$ so $(x+y)(x-y)$ is even. Therefore one of $(x+y)$ and $(x-y)$ must be even. However, since $2002/2 = 1001$ which is odd, one of $(x+y)$ and $(x-y)$ must be odd. The sum of an even and odd number is odd, which implies $(x+y)+(x-y)$ must be odd, but $(x+y)+(x-y) = 2x$ which is even. This is a contradiction. Therefore no integer solutions.

   (c) Let $n \in \mathbb{N}$, $n > 1$. If n is not prime then $2^n - 1$ is not prime. Hint: you may use the identity $(a^x - 1) = (a-1)(a^{x-1} + a^{x-2} + \cdots + a^1 + 1)$.
   If n is not prime, then n is composite and can be written as $n = \beta \times \gamma$ with $1 < \beta\,,\ \gamma < n$:

   $$2^{\gamma\beta} - 1 = (2^\gamma - 1)(2^{\gamma(\beta-1)} + 2^{\gamma(\beta-2)} + \cdots + 2^{\gamma 1} + 2^{\gamma 0})$$

   In the above, $\gamma > 1 \Rightarrow (2^\gamma - 1) > 1$ and $\beta > 1 \Rightarrow (2^{\gamma(\beta-1)} + 2^{\gamma(\beta-2)} + \cdots + 2^{\gamma 1} + 2^{\gamma 0}) < 2^n - 1$.
   Therefore $2^n - 1$ is composite.

2. Prove the following by induction. State base case, inductive hypothesis and inductive step explicitly.

   (a) For all $n \in \mathbb{N}$, $\sum_{i=1}^{n} (2i-1) = n^2$
   Base case when $n = 1 : (2 \times 1 - 1) = 1^2$
   Inductive hypothesis: Assume $\sum_{i=1}^{k} (2i-1) = k^2$
   Inductive step:

   $$\begin{aligned}\sum_{i=1}^{k+1} (2i-1) &= \underbrace{\sum_{i=1}^{k} (2i-1)}_{k^2 \text{ by IH}} + 2(k+1) - 1 \\ &= k^2 + 2k + 1 \\ &= (k+1)^2 \quad \square\end{aligned}$$

   (b) For all $n \in \mathbb{N}$, $n \geq 4$, $3^n \geq n^3$
   Base case when $n = 4 : 3^4 = 81 \geq 4^3 = 64$

Inductive hypothesis: Assume $n \geq 4$, $3^n \geq n^3$ for some $k \geq 4$
Inductive step:

$$\begin{aligned}
3^{k+1} &= 3^1 \times 3^k \\
&\geq k^3 + k^3 + k^3 \text{ (by IH)} \\
&\geq k^3 + 3k^2 + 3k^2 \text{ (since } k \geq 4) \\
&= k^3 + 3k^2 + k^2 + k^2 + k^2 \text{ (expanding the last term)} \\
&\geq k^3 + 3k^2 + 3k + 3k + 3k \text{ (since } k \geq 4) \\
&\geq k^3 + 3k^2 + 3k + 1 \text{ (since } k \geq 4) \\
&= (k+1)^3 \quad \square
\end{aligned}$$

(c) For all $n \in \mathbb{N}$, $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$
Base case when $n = 1 : \frac{1}{1(1+1)} = \frac{1}{1+1}$.
Inductive hypothesis: Assume $\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$ true for some k.
Inductive step:

$$\begin{aligned}
\sum_{i=1}^{k+1} \frac{1}{i(i+1)} &= \underbrace{\sum_{i=1}^{k} \frac{1}{i(i+1)}}_{\frac{k}{k+1} \text{ by IH}} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k(k+2)+1}{(k+1)(k+2)} \\
&= \frac{k^2+2k+1}{(k+1)(k+2)} \\
&= \frac{(k+1)(k+1)}{(k+1)(k+2)} \\
&= \frac{k+1}{k+2} \quad \square
\end{aligned}$$

3. Use the Euclidean Algorithm to find the greatest common divisor of 143 and 33. Use the Extended Euclidean Algorithm to find the Bezout coefficients x, y and all integer solutions to the equation $143x + 33y = gcd(143, 33)$.

$$\begin{aligned}
143 &= 4 \times 33 + 11 \Rightarrow 11 = 143 - 4 \times 33 \\
33 &= 3 \times 11 + 0
\end{aligned}$$

Therefore the $gcd(143, 33) = 11$ and $x = 1$, $y = -4$. Using the identity $lcm(a, b) = \frac{ab}{gcd(a,b)}$ we find that:

$$\begin{aligned}
143 \times \frac{33}{11} &= 33 \times \frac{143}{11} \\
143 \times 3 &= 33 \times 13 \\
143 \times 3 - 33 \times 13 &= 0 \\
143 \times 3k + 33 \times (-13k) &= 0
\end{aligned}$$

Adding the above with Bezout's identity:

$$143 \times (3k+1) + 33 \times (-13k-4) = 11$$

4. Prove the following by induction. State base case, inductive hypothesis and inductive step explicitly.

   (a) For all $n \in \mathbb{N}$, $n \geq 2$, $\prod_{i=2}^{n} \left(1 - \frac{1}{i^2}\right) = \frac{n+1}{2n}$.
   Base case when $n = 2 : (1 - \frac{1}{2^2}) = \frac{3}{4} \equiv \frac{2+1}{4}$.
   Inductive hypothesis: Assume $\prod_{i=2}^{k} \left(1 - \frac{1}{i^2}\right) = \frac{k+1}{2k}$ true for some k.
   Inductive step:

$$\begin{aligned}
\prod_{i=2}^{k+1}\left(1-\frac{1}{i^2}\right) &= \underbrace{\prod_{i=2}^{k}\left(1-\frac{1}{i^2}\right)}_{\frac{k+1}{2k}\text{ by IH}} \times \left(1-\frac{1}{(k+1)^2}\right) \\
&= \left(\frac{k+1}{2k}\right) \times \left(\frac{k^2+2k}{k^2+2k+1}\right) \\
&= \frac{k(k+1)(k+2)}{2k(k+1)(k+1)} \\
&= \frac{k+2}{2(k+1)} \quad \square
\end{aligned}$$

   (b) For all $n \in \mathbb{N}$, $7|(2^{n+2} + 3^{2n+1})$
   Base case when $n = 1 : 7|(2^3 + 3^3) \Rightarrow 7|35$
   Inductive hypothesis: Assume $7|(2^{k+2} + 3^{2k+1})$ for some k.
   Inductive step:

$$\begin{aligned}
2^{k+3} + 3^{2k+2} &= (9-7) \times 2^{k+3} + 9 \times 3^{2k+1} \\
&= 9(\underbrace{2^{k+2} + 3^{2k+1}}_{7|(2^{k+2}+3^{2k+1})\text{ by IH}}) - \underbrace{7 \times 2^{k+2}}_{7|7\times 2^{k+2}}
\end{aligned}$$

   Therefore $9(2^{k+2} + 3^{2k+1}) - 7 \times 2^{k+2}$ is divisible by 7 $\square$

5. Verify whether the following functions define bijections.

   (a) Let $A = \{x \in \mathbb{R} : x \neq -2\}$ and $B = \{x \in \mathbb{R} : x \neq 1\}$ Is the function $f : A \to B$ defined by $f(x) = \frac{x-2}{x+2}$ (i) injective (ii) surjective and (iii) bijective? Prove or disprove.

      i. $f(x_1) = f(x_2) \Rightarrow \frac{x_1-2}{x_1+2} = \frac{x_2-2}{x_2+2} \Rightarrow (x_1-2)(x_2+2) = (x_2-2)(x_1+2) \Rightarrow x_1x_2 - 2x_2 + 2x_1 - 4 = x_1x_2 + 2x_2 - 2x_1 - 4 \Rightarrow -2x_2 + 2x_1 = 2x_2 - 2x_1 \Rightarrow 4x_1 = 4x_2 \Rightarrow x_1 = x_2$. Therefore $f(x)$ is injective.

      ii. $y = \frac{x-2}{x+2} \Rightarrow y(x+2) = x - 2 \Rightarrow yx - x + 2y = -2 \Rightarrow x(y-1) + 2y = -2 \Rightarrow x(y-1) = -2(1+y) \Rightarrow x = \frac{-2(1+y)}{y-1}$. Given that $1 \notin B$, taking $f(\frac{-2-2x}{x-1}) = y$. Therefore $f(x)$ is surjective.

   (b) Is the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ (i) injective (ii) surjective and (iii) bijective? What if $f : \mathbb{Q} \to \mathbb{Q}$ or $f : \{-1, 0, 2\} \to \{-1, 0, 8\}$? Prove or disprove.

      $\Rightarrow$ $f(x_1) = f(x_2) \Rightarrow x_1^3 = x_2^3 \Rightarrow x_1 = x_2$.
      $\Rightarrow$ $f^{-1}(x) = x^{1/3}$.

      We can now verify for each domain and codomain below:

      i. $f : \mathbb{R} \to \mathbb{R}$.
      Bijective. Both derivations above hold.

ii. $f : \mathbb{Q} \to \mathbb{Q}$.
Injective, but not surjective. The cubed root is not always rational. For example, $\sqrt[3]{2} \notin \mathbb{Q}$.

iii. $f : \{-1, 0, 2\} \to \{-1, 0, 8\}$.
Bijective. Both derivations above hold.

7. Construct a bijection $f : [a, b) \to [0, 1)$. Show that f is one-to-one and onto.

$$f(x) = \frac{x - a}{b - a}$$

We need $0 \leq a < b$

(a) It is straightforward to verify that $f(x_1) = f(x_2) \Rightarrow \frac{x_1 - a}{b-a} = \frac{x_2 - a}{b-a} \Rightarrow (x_1 - a)(b - a) = (x_2 - a)(b - a) \Rightarrow x_1 - a = x_2 - a \Rightarrow x_1 = x_2$. Therefore $f(x)$ is one-to-one.

(b) $y = \frac{x-a}{b-a} \Rightarrow y(b - a) = (x - a) \Rightarrow x = y(b - a) + a \Rightarrow f(x) = \frac{x(b-a)+a-a}{b-a} = x \Rightarrow f(x)$ is onto.